

VECTOR BUNDLES ON HIRZEBRUCH SURFACES WHOSE TWISTS BY A NON-AMPLE LINE BUNDLE HAVE NATURAL COHOMOLOGY

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ABSTRACT. Here we study vector bundles E on the Hirzebruch surface F_e such that their twists by a spanned, but not ample, line bundle $M = \mathcal{O}_{F_e}(h + ef)$ have natural cohomology, i.e. $h^0(F_e, E(tM)) > 0$ implies $h^1(F_e, E(tM)) = 0$.

1. INTRODUCTION

Let F_e , $e > 0$, denote the Hirzebruch surface with a section with self-intersection $-e$. For any $L \in \text{Pic}(F_e)$ and any vector bundle E on F_e we will say that E has property $\mathcal{L}\mathcal{L}$ (resp. \mathcal{L}) with respect to L if $h^1(F_e, E \otimes L^{\otimes m}) = 0$ for all $m \in \mathbb{Z}$ (resp. for all $m \in \mathbb{Z}$ such that $h^0(F_e, E \otimes L^{\otimes m}) \neq 0$). We think that property \mathcal{L} is nicer for reasonable L . We take as a basis of $\text{Pic}(F_e) \cong \mathbb{Z}^2$ a fiber f of the ruling $\pi : F_e \rightarrow \mathbf{P}^1$ and the section h of π with negative self-intersection. Thus $h^2 = -e$, $h \cdot f = 1$ and $f^2 = 0$. We have $\omega_{F_e} \cong \mathcal{O}_{F_e}(-2h - (e+2)f)$. $\mathcal{O}_{F_e}(\alpha h + \beta f)$ is spanned (resp. ample) if and only if $\alpha \geq 0$ and $\beta \geq \alpha e$ (resp. $\alpha > 0$ and $\beta > \alpha e$). The Leray spectral sequence of π and Serre duality give that $h^1(F_e, \mathcal{O}_{F_e}(\gamma h + \delta f)) = 0$ if and only if either $\gamma \geq 0$ and $\delta \geq e\gamma - 1$ or $\gamma = -1$ or $\gamma \leq -2$ and $-\delta - e - 2 \geq e(-\gamma - 2) - 1$ (i.e. $\delta \leq e\gamma + e - 1$). We consider as the test line bundle the spanned, but not ample, line bundle $M := \mathcal{O}_{F_e}(h + ef)$. Notice that the linear system $|M^{\otimes 2}|$ contains the sum of the effective divisor h and the ample divisor $h + 2ef$. Thus for every vector bundle E on F_e there is an integer $m_0(E)$ such that $h^0(F_e, E \otimes M^{\otimes m}) \neq 0$ for all $m \geq m_0(E)$. We will see that property $\mathcal{L}\mathcal{L}$ is too strong and not interesting (see Remarks 1 and 2). We stress the property \mathcal{L} with respect to M is quite different from similar looking properties (e.g. natural cohomology) with respect to an ample line bundle. (see Remarks 2 and 3 for the rank 1 case). Obviously, properties \mathcal{L} and $\mathcal{L}\mathcal{L}$ may be stated for arbitrary projective varieties. In dimension $n \geq 3$, one need to choose between vanishing of h^1 or vanishing of all h^i , $1 \leq i \leq n - 1$. We considered here the example (F_e, M) , because it is geometrically significant. Indeed, let ϕ_M denote the morphism associated to the base point free linear system $|M|$. If $e = 1$ the morphism ϕ_M is the blowing up $F_1 \rightarrow \mathbf{P}^2$. If $e \geq 2$, then $\phi_M : F_e \rightarrow \mathbf{P}^{e+1}$ contracts h and its image is a cone over the rational normal curve of \mathbf{P}^e . Moreover, for any spanned and non-trivial line bundle L on F_e there is an effective divisor D such that $L \cong M(D)$. For any spanned, but not ample line bundle A on F_e there is an integer $c \geq 0$ such that $A \cong M^{\otimes c}$. We prove the following results.

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Theorem 1. Fix integers $e \geq 1$, $r \geq 1$, u, v such that $v \leq e(u - r + 1) - 2$. Then there is no rank r vector bundle E on F_e with property \mathcal{L} with respect to M such that $c_1(E) = \mathcal{O}_{F_e}(uh + vf)$.

Theorem 2. Fix integers e, m, u, v such that $e \geq 1$, and $v \geq e(u - 1) - 1$ and $m \geq 0$. Set $\tilde{a} := \sum_{i=0}^{u+2m-2} v + 2m - 1 - ie$ and $\tilde{b} := \sum_{i=0}^{u+2m-1} v + 2m - ie$. Fix any integer s such that $\tilde{a} \leq s \leq \tilde{b}$. Then there exists a rank 2 vector bundle E on F_e with property \mathcal{L} with respect to M such that $c_1(E) = \mathcal{O}_{F_e}(uh + vf)$ and $c_2(E) := s - e(u + m - 1) + (1 - m)(v + em)$. Set $R := \mathcal{O}_{F_e}(h + (e + 1)f)$ and assume $m = 0$, $u \geq 3$, and $v < 2eu$. Then we may find E as above which is R -stable in the sense of Mumford-Takemoto and (under the additional condition $v \leq 2eu - 3$) such that $N \cdot M < c_1(E) \cdot M/2$ for all rank 1 subsheaves N of E .

The case “ $r = 1$ ” of Theorem 1 is obviously true (use the cohomology of line bundles on F_e , i.e. Remark 2 below). In this case the converse is true, i.e. $\mathcal{O}_{F_e}(uh + vf)$ has property \mathcal{L} with respect to M if and only if $v \geq eu - 1$ (Remark 2). We were surprised that for $r \geq 2$ there is no way to overcome this c_1 -obstruction.

The assumptions of the last part of Theorem 2 may be relaxed and instead of R we may take an arbitrary ample divisor H . An interesting offshot of our proof of Theorem 2 is that our examples are given by an extension (4) and all locally free sheaves fitting in (4) have property \mathcal{L} with respect to M and (under the additional conditions listed in Theorem 2) are R -stable and $N \cdot M < c_1(E) \cdot M/2$ for all rank 1 subsheaves N of E .

In the case of direct sums of line bundles we will prove the following result.

Proposition 1. Fix integers $e \geq 1$, $r \geq 2$ and $L_i \in \text{Pic}(F_e)$, $1 \leq i \leq r$, say $L_i \cong \mathcal{O}_{F_e}(u_i h + v_i f)$. Set $E := L_1 \oplus \cdots \oplus L_r$. Up to a permutation of the factors of E we may assume $u_1 \geq \cdots \geq u_r$ and that if $u_i = u_j$ for some $i < j$, then $v_i \geq v_j$. Set $m := -u_1$ if $v_1 \geq eu_1$ and $m := -u_1 + 1$ if $v_1 = eu_1 - 1$. The vector bundle E has property \mathcal{L} with respect to M if and only if $v_i \geq eu_i - 1$ for all i , and for each $i \in \{2, \dots, r\}$ either $u_i - m \geq -1$ or $-1 \leq v_i - eu_i \leq e - 1$.

We raise the following question.

Question 1. Assume $e = 1$ or $e = 2$. Is it possible to describe all invariants r, c_1, c_2 of vector bundles on F_e with property \mathcal{L} with respect to M ?

2. THE PROOFS

For any sheaf F we will often write $F(mM)$ instead of $F \otimes M^{\otimes m}$.

Remark 1. The line bundle $\mathcal{O}_{F_e}(ch + df)$ is ample if and only if $c > 0$ and $d > ec$. Hence any ample line bundle is spanned. Assume that $H := \mathcal{O}_{F_e}(ch + df)$ is ample. The cohomology of line bundles on F_e shows that for every $t \in \mathbb{Z}$ the line bundle $H^{\otimes t}$ has property \mathcal{L} with respect to H . Hence \mathcal{O}_{F_e} has property \mathcal{L} with respect to any ample line bundle. Set $H' := \mathcal{O}_{F_e}(ch + (d + 2)f)$. Taking $m_t := -tc$ we see that no $H^{\otimes t}$, $t > 0$, has property \mathcal{L} with respect to the ample line bundle H' . Taking $m_t := -tc$ we see that no $H^{\otimes t}$, $t > 0$, has property \mathcal{L} with respect to M .

Remark 2. Here we study properties \mathcal{L} and \mathcal{L} with respect to M for line bundles on F_e . Fix $L \in \text{Pic}(F_e)$, say $L \cong \mathcal{O}_{F_e}(uh + vf)$. First assume $v \geq eu$. We have $h^0(F_e, L(xM)) > 0$ if and only if $x \geq -u$. Since $h^1(F_e, \mathcal{O}_{F_e}(ch + df)) = 0$ if $c \geq 0$ and $d \geq ec$, L has property \mathcal{L} with respect to M . Now assume $v < eu$. We have

$h^0(F_e, L(xM)) > 0$ if and only if $ex \geq -v$. Since $h^1(F_e, \mathcal{O}_{F_e}(ch + df)) = 0$ if $c \geq 0$ if and only if $d \geq ec - 1$, we get that L has property \mathcal{L} with respect to M if and only if $v \geq eu - 1$. Take $m := -u$. If $v = eu$, then we saw in the introduction that L has property \mathcal{L} with respect to M if and only if $e = 1$. Notice that $h^1(F_e, \mathcal{O}_{F_e}((u-x)h + (v-ex)f)) = h^1(F_e, \mathcal{O}_{F_e}((x-u-2)h + (ex-v-e-2))) > 0$ when $x \geq -u-2$ if and only if $-eu-2e \leq -v-e-1$, i.e. if and only if $v \leq eu+e-1$. Notice that $h^1(F_e, \mathcal{O}_{F_e}((u+x)h + (v+ex)f)) = 0$ for $x \geq -u$ if and only if $v \geq eu-1$. Notice that $h^1(F_e, \mathcal{O}_{F_e}(-h + (v-eu-e)f)) = 0$ for every $v \in \mathbb{Z}$. Hence L has property \mathcal{L} with respect to M if and only if $eu-1 \leq v \leq eu+e-1$.

Remark 3. Here we look at property \mathcal{L} with respect to the ample line bundle $R := \mathcal{O}_{F_e}(h + (e+1)f)$ for line bundles on F_e . Fix $L \in \text{Pic}(F_e)$, say $L \cong \mathcal{O}_{F_e}(uh + vf)$. We have $h^0(F_e, L(xR)) = 0$ if and only if $x \geq -u$ and $x(e+1) \geq -v$. We immediately see that if $v \geq (e+1)u$, then L has property \mathcal{L} with respect to M . Now assume $v < (e+1)u$. Set $y := \lceil -v/(e+1) \rceil$. We have $h^0(F_e, L(xR)) > 0$ if and only if $x \geq y$. Fix an integer $x \geq y$. Since $u+x \geq u+y \geq 0$, $h^1(F_e, \mathcal{O}_{F_e}(u+x)h + (v+(e+1)x)f)) > 0$ if and only if $v + (e+1)x \leq eu + ex - 2$. The strongest condition is obtained when $x = y$. We get that L has property α with respect to R if and only if either $v \geq (e+1)u$ or $v + ey \geq eu - 1$, where $y := \lceil -v/(e+1) \rceil$.

Remark 4. $E_1 \oplus E_2$ has property \mathcal{L} with respect to L if and only if both E_1 and E_2 have property \mathcal{L} with respect to L . If $E_1 \oplus E_2$ has property \mathcal{L} with respect to L , then the same is true for E_1 and E_2 . Now we check that the converse is not true. Both \mathcal{O}_{F_e} and $\mathcal{O}_{F_e}(-2h + (-e+4)f)$ have property \mathcal{L} with respect to M (Remark 2). Since $h^1(F_e, \mathcal{O}_{F_e}(-2h + (-e+4)f)) = h^1(F_e, \mathcal{O}_{F_e}(-2f)) = 1$, $\mathcal{O}_{F_e} \oplus \mathcal{O}_{F_e}(-2h + (-e+4)f)$ has not property \mathcal{L} with respect to M .

Remark 5. The definition of property \mathcal{L} may be given for an arbitrary torsion free sheaf, but not much may be said in the general case. Here we look at the rank 1 case, because we will need it in the proofs of Theorems 1 and 2. Let A be a rank 1 torsion free sheaf on F_e . Hence $A \cong \mathcal{I}_Z(uh + vf)$ for some zero-dimensional scheme Z and some integers u, v . Since Z is zero-dimensional, $h^1(F_e, \mathcal{O}_{F_e}((u+t)h + (v+et)f)) \leq h^1(F_e, \mathcal{I}_Z((u+t)h + (v+et)f))$ for all $t \in \mathbb{Z}$. Taking $t \gg 0$ we see that if A has property \mathcal{L} with respect to M , then $v \geq eu - 1$. When $v \geq eu - 1$, for a general Z (in the following sense) A has property \mathcal{L} with respect to M for the following reason. Fix an integer $z > 0$. Since F_e is a smooth surface, the Hilbert scheme $\text{Hilb}^z(F_e)$ of all length z zero-dimensional subschemes of F_e is irreducible and of dimension $2z$ ([2]). Take a general $S \in \text{Hilb}^z(F_e)$, i.e. take z general points of F_e . Since $h^0(F_e, \mathcal{I}_S \otimes L) = \max\{0, h^0(F_e, L) - z\}$ for every $L \in \text{Pic}(F_e)$, it is easy to check that if $v \geq eu - 1$, then $\mathcal{I}_S(uh + vf)$ has property \mathcal{L} with respect to M . Now take $v = eu - 1$, any integer $z > 0$ and any zero-dimensional length z subscheme B of h . Twisting with $(-u+1)M$ we see that $\mathcal{I}_B(uh, (eu-1)f)$ has not property \mathcal{L} with respect to M . Now assume $v > eu$. Take a zero-dimensional length $z \geq 2$ scheme W of a fiber of π . Twisting with $-uM$. We see that $\mathcal{I}_W(uh + vf)$ has not property \mathcal{L} with respect to M . If $z \geq 3$ and $v = eu$, twisting with $(-u+1)M$ and using the same W we get a sheaf without property \mathcal{L} with respect to M .

Property \mathcal{L} with respect to M has the following open property.

Proposition 2. *Let $\{E_t\}_{t \in T}$ be a flat family of vector bundles on F_e parametrized by an integral variety T . Assume the existence of $s \in T$ such that E_s has property*

\mathcal{L} with respect to M . Then there exists an open neighborhood U of s in T such that E_t has property \mathcal{L} with respect to M for all $t \in U$.

Proof. Let m be the minimal integer such that $h^0(F, e, E_s(mM)) > 0$. Thus $h^1(F_e, E_s(xM)) = 0$ for all $x \geq m$. By semicontinuity there is an open neighborhood V of s in T such that $h^0(F_e, E_t((m-1)M)) = 0$ for all $t \in V$. By semicontinuity for every integer $x \geq m$ there is an open neighborhood V_x of s in T such that $h^1(F_e, E_t(xM)) = 0$ for all $t \in V_x$. Fix an irreducible $D \in |M|$. Hence $D \cong \mathbf{P}^1$. Since $D^2 > 0$, there is an integer a such that $h^1(D, E_s(aM)|D) = 0$. By semicontinuity there is an open neighborhood V of s in T such that $h^1(D, E_t(aM)|D) = 0$ for every $t \in V$. Since $D^2 > 0$, $h^1(D, E_t(xM)|D) = 0$ for every $t \in V$ and every integer $x \geq a$. Fix an integer $x \geq a$. From the exact sequence

$$0 \rightarrow E_t((x-1)M) \rightarrow E_t(xM) \rightarrow E_t(xM)|D \rightarrow 0$$

we get that if $h^1(F_e, E_t((x-1)M)) = 0$, then $h^1(F_e, E_t(xM)) = 0$. Hence we may take $U := V \cap \bigcap_{x=m}^{\max\{a, m\}} V_x$. \square

Proof of Proposition 1. If E has property \mathcal{L} with respect to M , then each L_i has property \mathcal{L} with respect to M (Remark 4) and hence $v_i \geq eu_i - 1$ for all i . Now we assume $v_i \geq eu_i - 1$ for all i . Notice that m is the minimal integer t such that $h^0(F_e, E(t)) \neq 0$. Since L_1 has property \mathcal{L} with respect to M , E has property \mathcal{L} with respect to E if and only if $h^1(F_e, L_i(tM)) = 0$ for all $t \geq m$ and all $i = 2, \dots, r$. If $u_i - m \geq -1$, then $h^1(F_e, L_i(tM)) = 0$ for all $t \geq m$ because $v_i \geq eu_i - 1$. Now assume $u_i - m \leq -2$. We get $h^1(F_e, L_i(tM)) = 0$ for any $t \geq m$ if and only if $-1 \leq v_i - eu_i \leq e - 1$. \square

Here we discuss the set-up for the rank 2 case. Consider an exact sequence

$$(1) \quad 0 \rightarrow \mathcal{O}_{F_e}(D) \rightarrow E(mM) \rightarrow \mathcal{I}_Z(c_1 + 2mM - D) \rightarrow 0$$

in which Z is a zero-dimensional scheme with length s and either $D = 0$ or $D = h$ or $D \in |zf|$ for some $1 \leq z \leq e$ or $e \geq 2$ and $D \in |h + wf|$ for some $1 \leq w \leq e - 1$. We have $c_1(E(mM)) = c_1 + 2nm$ and $c_2(E(mM)) = s + D \cdot c_1 + 2mM \cdot D - D^2$. Thus $c_1(E) = c_1$ and $c_2(E) = c_2$ by the choice of s ([3], Lemma 2.1). Each E fitting in (1) is torsion free. To have some locally free E fitting in (1) a necessary condition is that Z is a locally complete intersection. Notice that $h^1(F_e, \mathcal{O}_{F_e}(D)(-M)) = 0$ if h is not a component of D . Hence a sufficient condition to have $h^0(F_e, E(mM)) > 0$ and $h^0(F_e, E((m-1)M)) = 0$ is the equality

$$(2) \quad h^0(F_e, \mathcal{I}_Z(c_1 + (2m-1)M - D)) = 0$$

and (2) is a necessary condition if h is not a component of D . Assume that Z is a locally a complete intersection. The Cayley-Bacharach condition associated to (1) is satisfied if

$$(3) \quad h^0(F_e, \mathcal{I}_{Z'}(c_1 + 2mM - 2D - 2h - (e+2)f)) = 0$$

for every length $s-1$ closed subscheme of s ([1]). This condition is satisfied if $h^0(F_e, \mathcal{I}_Z(c_1 + (2m-1)M - D)) = 0$, $Z_{red} \cap h = \emptyset$ and no connected component of Z is tangent to a fiber of the fiber of π , because the line bundle $\omega_{F_e}^*(-M) = \mathcal{O}_{F_e}(h + 2f)$ is base point free outside h and the morphism associated to $|f|$ is the ruling; if $e = 12$, then (3) is satisfied if (2) is satisfied, because $\mathcal{O}_{F_1}(h + 2f)$ is

very ample; if $e = 2$ it is sufficient to assume $Z_{red} \cap h = \emptyset$, because the morphism associated to $\mathcal{O}_{F_2}(h + 2f)$ is an embedding outside h .

Proof of Theorem 1 for $r \leq 2$. If $r = 1$, then use Remark 2. Assume the existence of a rank two vector bundle E with property \mathcal{L} with respect to M and $c_1(E) = \mathcal{O}_{F_e}(uh + vf)$. Let m be the first integer such that $h^0(F_e, E(mM)) > 0$. We get an exact sequence (1) with $D \in |\mathcal{O}_{F_e}(xh + yf)|$ with the convention $(x, y) = (0, 0)$ if $D = \emptyset$. Hence either $(x, y) = (0, 0)$ or $(x, y) = (1, 0)$ or $x = 0$ and $1 \leq y \leq e$ or $e \geq 2$, $x = 1$, and $1 \leq y \leq e - 1$. Since $h^2(F_e, M^{\otimes z}(D)) = 0$ for all $z \geq 0$, (1) and property \mathcal{L} for E imply $h^1(F_e, \mathcal{I}_Z((u + 2m - x + z)h + (v + 2me - y + ze)f)) = 0$ for all $z \geq 0$. As in Remark 5 we see that when $z \gg 0$ the last equality implies $v - y \geq e(u - x) - 1$. If $v \leq e(u - 1) - 2$ the last inequality is not satisfied for any choice of the pair (x, y) in the previous list. \square

Proof of Theorem 2. Fix a general $S \subset F_e$ such that $\sharp(S) = s$. Let E be any torsion free sheaf fitting in the following exact sequence:

$$(4) \quad 0 \rightarrow \mathcal{O}_{F_e}((1 - m)h - emf) \rightarrow E \rightarrow \mathcal{I}_S((u + m - 1)h + (v + em)f) \rightarrow 0$$

We have $c_1(E) = \mathcal{O}_{F_e}(uh + vf)$ and $c_2(E) = s - e(u + m - 1) + (1 - m)(v + em)$. By construction $h^0(F_e, E(mM)) \neq 0$. We have $h^0(F_e, E((m - 1)M)) = 0$. If $h^0(F_e, \mathcal{I}_S((u + 2m - 2)h + (v + 2em - e)f)) = 0$. Since S is general, $h^0(F_e, \mathcal{I}_S((u + 2m - 2)h + (v + 2em - e)f)) = 0$ if and only if

$$(5) \quad h^0(F_e, \mathcal{O}_{F_e}((u + 2m - 2)h + (v + 2em - e)f)) \leq s$$

Since S is general, every subset of it is general. Hence to check the Cayley-Bacharach condition and hence show the local freeness of a general E given by the extension (5) it is sufficient to prove check the following inequality:

$$(6) \quad h^0(F_e, \mathcal{O}_{F_e}((u + 2m - 5)h + (v + 2em - 2e - 2)f)) \leq s - 1$$

This is true, because we assumed $s \geq \tilde{a}$ and $\tilde{a} > h^0(F_e, \mathcal{O}_{F_e}((u + 2m - 5)h + (v + 2em - 2e - 2)f))$. Hence a general E fitting in the extension (5) is locally free. Since $\mathcal{O}_{F_e}(3h + e + 2)$ has a subsheaf the very ample line bundle $\mathcal{O}_{F_e}(h + e + 2)$, (6) is satisfied if (5) is satisfied. The generality of S implies that $h^1(F_e, \mathcal{I}_S((u + m - 1 + t)h + (v + em + et)f)) = 0$ if and only if $h^1(F_e, \mathcal{O}_{F_e}((u + m - 1 + t)h + (v + em + et)f)) = 0$ and $h^0(F_e, \mathcal{O}_{F_e}((u + m - 1 + t)h + (v + em + et)f)) \geq s$. Notice that $\tilde{a} = h^0(F_e, \mathcal{O}_{F_e}((u + 2m - 2)h + (v + 2me - e)f))$ and $\tilde{b} = h^0(F_e, \mathcal{O}_{F_e}((u + 2m - 1)h + (v + 2me)f))$. Since $h^1(F_e, (u + m - 1 + t)u + (v + me + te)f)) = 0$ for all $t \geq 0$, $\tilde{a} \leq s \leq \tilde{b}$ and S is general, any sheaf E in (4) has property \mathcal{L} with respect to M . Since a general extension (4) has locally free middle term E , the proof of the first part of Theorem 2 is over. Now assume $m = 0$, $u \geq 3$, $v < 2eu$, and that E is not R -stable, i.e. assume the existence of $N \in \text{Pic}(F_e)$ such that $N \cdot R \geq c_1(E) \cdot R/2$ and an inclusion $j : N \rightarrow E$; here to have N locally free we use that E is reflexive. Since $m = 0$ and $u \geq 3$, $c_1(E) \cdot R > 2(\mathcal{O}_{F_e}(h)) \cdot R$. Hence j induces a non-zero map $N \rightarrow \mathcal{I}_S((u - 1)h + vf)$. Any non-zero map $N \rightarrow \mathcal{O}_{F_e}((u - 1)h + vf)$ is associated to a unique non-negative divisor $\Delta \in |\mathcal{O}_{F_e}((u - 1)h + vf) \otimes N^*|$. Since j factors through $\mathcal{I}_S((u - 1)h + vf)$, $h^0(F_e, \mathcal{I}_S(\Delta)) > 0$. We fixed R and the integers m, u, v . There are only finitely many possibilities for the line bundle $\mathcal{O}_{F_e}(\Delta)$. Since S is general, we get $h^0(F_e, \mathcal{O}_{F_e}(\Delta)) \geq s$. Write $N = \mathcal{O}_{F_e}(\gamma h + \delta f)$ for some integers

γ, δ . The inequality $N \cdot R \geq c_1(E) \cdot R/2$ is equivalent to the inequality

$$(7) \quad 2\gamma + 2\delta \geq u + v$$

We have $\mathcal{O}_{F_e}(\Delta) = \mathcal{O}_{F_e}((u-1-\gamma)h + (v-\delta)f)$. Since $h^0(F_e, \mathcal{O}_{F_e}(\Delta)) \geq s$ and $s \leq \tilde{b} = h^0(F_e, \mathcal{O}_{F_e}((u-1)h + (v)f))$, either $\gamma \leq 0$ or $\delta \leq 0$. Since Δ is effective, we also have $\gamma \leq u-1$ and $\delta \leq v$. First assume $\delta \leq 0$. Hence $\gamma \geq (u+v)/2$. Since $\gamma \leq u-1$, we get $v \leq u-2$. Since $v \geq eu - e$, we get a contradiction. Now assume $\gamma \leq 0$. We get $\delta \geq (u+v)/2$. Consider the exact sequence

$$(8) \quad 0 \rightarrow N \rightarrow E \rightarrow \text{Coker}(j) \rightarrow 0$$

Notice that $\text{Coker}(j)^{**} \cong \mathcal{O}_{F_e}((u-\gamma)h + (v-\delta)f)$. Since $\gamma \leq 0$, $\delta \geq (u+v)/2$, and $v < 2eu$, we have $v-\delta \leq e(u-\gamma)-2$. In Remark 5 we checked that $h^1(F_e, \text{Coker}(j)(tM)) > 0$ for $t \gg 0$. Since $h^2(F_e, L(tM)) = 0$ for $t \gg 0$ and any $L \in \text{Pic}(F_e)$, the exact sequence (8) gives that E has not property \mathcal{L} with respect to M , contradicting the already proved part of Theorem 2. If instead of R we use M for the intersection product, instead of (7) we only have the inequality $2\delta \geq v$. Everything works in the same way with only minor numerical modifications. \square

Remark 6. There are at least 2 well-known and related ways to obtain rank $r \geq 3$ vector bundles as extensions. Instead of (1) we may take the exact sequence

$$(9) \quad 0 \rightarrow \oplus_{i=1}^{r-1} \mathcal{O}_{F_e}(D_i - m_i M) \rightarrow \mathcal{I}_Z(uh + vf) \rightarrow 0$$

In [4], proof of Theorem 5.1.6, the following extension is used:

$$(10) \quad 0 \rightarrow L_1 \rightarrow E \rightarrow \oplus_{i=2}^r \mathcal{I}_{Z_i}(u_i h + v_i f) \rightarrow 0$$

The latter extension was behind the proof of Proposition 1. Both extensions can give several examples of vector bundles with or without property \mathcal{L} with respect to M . To prove Theorem 1 we will use iterated extensions, i.e. increasing filtrations E_i , $1 \leq i \leq r$, of E such that E_1 is a line bundle, $E_r = E$ and each E_i/E_{i-1} is a rank 1 torsion free sheaf.

Proof of Theorem 1 for $r \geq 3$. Assume the existence of a rank r vector bundle E with property \mathcal{L} with respect to M and $c_1(E) = \mathcal{O}_{F_e}(uh + vf)$. Let m_1 be the first integer such that $h^0(F_e, E(m_1 M)) > 0$. Fix a general $\sigma \in H^0(F_e, E(m_1 M))$. Since $h^0(F_e, E((m_1 - 1)M)) = 0$, σ induces an exact sequence

$$(11) \quad 0 \rightarrow \mathcal{O}_{F_e}(-m_1 M + D_1) \rightarrow E \rightarrow G_1 \rightarrow 0$$

with F_1 torsion free, D_1 of type (x_1, y_1) and either $(x_1, y_1) = (0, 0)$ or $(x_1, y_1) = (1, 0)$ or $x_1 = 0$ and $1 \leq y_1 \leq e$ or $e \geq 2$, $x_1 = 1$, and $1 \leq y_1 \leq e - 1$. Notice that $c_1(G_1) = \mathcal{O}_{F_e}((u + m_1 - x_1)h + (v + em_1 - y_1)f)$. Set $E_1 := \mathcal{O}_{F_e}(-m_1 M + D_1)$. Since $h^2(F_e, \mathcal{O}_{F_e}((t - m_1)D + D_1)) = 0$ for all $t \geq m_1$, property \mathcal{L} for m with respect to M implies $h^1(F_e, F_1(tM)) = 0$ for all $t \geq m_1$. Let m_2 be the first integer such that $m_2 \geq m_1$ and $h^0(F_e, F_1(m_2 M)) > 0$. A non-zero section of $H^0(F_e, G_1(m_2 M))$ induces an exact sequence

$$(12) \quad 0 \rightarrow \mathcal{I}_{Z_1}(-m_2 M + D_2) \rightarrow G_1 \rightarrow G_2 \rightarrow 0$$

with \mathcal{I}_{Z_1} zero-dimensional, G_2 torsion free and D_2 an effective divisor of type (x_2, y_2) and either $(x_2, y_2) = (0, 0)$ or $(x_2, y_2) = (1, 0)$ or $x_2 = 0$ and $1 \leq y_2 \leq e$ or $e \geq 2$, $x_2 = 1$, and $1 \leq y_2 \leq e - 1$. Here we cannot claim that $Z_1 = \emptyset$, because G_1 is not assumed to be locally free. Notice that $c_1(G_2) = \mathcal{O}_{F_e}((u + m_1 + m_2 - x_1 - x_2)h + (v + em_1 + em_2 - y_1 - y_2)f)$. Since Z_1 is zero-dimensional,

$h^2(F_e, \mathcal{I}_{Z_1} \otimes L) = h^2(F_e, L)$ for every $L \in \text{Pic}(F_e)$. Hence as in the first step we get $h^1(F_e, G_2(tM)) = 0$ for all $t \geq m_2$. If $r = 3$, we are done as in the proof of the case $r = 2$. If $r \geq 4$, we iterate the last step $r - 3$ times. \square

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